



# Positive Solution for Three-Point Boundary Value Problems with Sign Changing Nonlinearities

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**Abstract**—By constructing available operators, some new existence theorems of positive solutions are obtained for a class of three-point boundary value problems

$$\begin{aligned}u'' + \lambda h(t)f(t, u) &= 0, & 0 \leq t \leq 1, \\u(0) - \beta u'(0) &= 0, & u(1) = \alpha u(\eta),\end{aligned}$$

where  $f$  is allowed to change sign,  $\eta \in (0, 1)$ . The associated Green's function for the above problem is also given. © 2004 Elsevier Ltd. All rights reserved.

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## 1. INTRODUCTION

The multipoint boundary value problems of differential equations or difference equations arise in a variety of different applied mathematics and physics. The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1] motivated by the work of Bitsadze [2] on nonlocal linear elliptic boundary problems. Since then, nonlinear multipoint boundary value problems have been studied by several authors, for example, see [3–6] and their references. In [7], the three-point boundary value problem

$$u''(t) + a(t)f(u) = 0, \quad 0 \leq t \leq 1, \quad (1.1)$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \quad (1.2)$$

is considered, where  $\alpha > 0$ ,  $\eta \in (0, 1)$ ,  $\alpha\eta < 1$ ,  $a \in C([0, 1], [0, \infty))$ , and  $f \in C([0, \infty), [0, \infty))$  is suplinear or sublinear. Reference [7] showed the existence of at least one positive solution for

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BVP (1.1),(1.2) by applying the Krasnosel'skii fixed-point theorem on the condition that  $a(t) \geq 0$ ,  $f(u) \geq 0$ . Recently, Raffoul [8] generalized BVP (1.1),(1.2) to

$$u''(t) + \lambda a(t)f(u) = 0, \quad 0 \leq t \leq 1, \quad (1.3)$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \quad (1.4)$$

where  $0 < \eta < 1$ ,  $0 < \alpha < 1/\eta$ ,  $a \in C([0, 1], [0, \infty))$ , and  $f \in C([0, \infty), [0, \infty))$ . In [8], the author obtained the existence of at least one positive solution for BVP (1.3),(1.4) when  $f$  is not required to be either sublinear or superlinear. All the above works were done under the assumption that the nonlinear term is nonnegative due to applying the concavity of solutions in the proofs.

In this paper, we consider the following three-point boundary value problem:

$$u''(t) + \lambda h(t)f(t, u) = 0, \quad 0 \leq t \leq 1, \quad (1.5)$$

$$u(0) - \beta u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad (1.6)$$

where the nonlinear term  $f$  is continuous and is allowed to change sign. By constructing available operators, we combine the method of lower solution with the method of topology degree and show that BVP (1.5),(1.6) has at least one positive solution with certain growth conditions imposed on  $f$ . In this way we removed the usual restriction on  $f$ .

## 2. PRELIMINARY

Before the statement of our main results, we give some lemmas which are needed later.

LEMMA 2.1. Suppose that  $(1 - \alpha\eta) + \beta(1 - \alpha) \neq 0$ ,  $y(t) \in C[0, 1]$ , then BVP

$$u'' + y(t) = 0, \quad 0 \leq t \leq 1, \quad (2.1)$$

$$u(0) - \beta u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad (2.2)$$

has a unique solution

$$\begin{aligned} u(t) = & - \int_0^t (t-s)y(s) ds + \frac{t+\beta}{(1-\alpha\eta) + \beta(1-\alpha)} \int_0^1 (1-s)y(s) ds \\ & - \frac{\alpha(t+\beta)}{(1-\alpha\eta) + \beta(1-\alpha)} \int_0^\eta (\eta-s)y(s) ds. \end{aligned} \quad (2.3)$$

PROOF. Integrating both sides of (2.1) on  $[0, t]$ , we have

$$u'(t) = - \int_0^t y(s) ds + u'(0). \quad (2.4)$$

Again integrating (2.4) from 0 to  $t$ ,

$$u(t) = - \int_0^t (t-s)y(s) ds + u'(0)t + u(0). \quad (2.5)$$

By (2.2), we get

$$u'(0) = \frac{1}{(1-\alpha\eta) + \beta(1-\alpha)} \int_0^1 (1-s)y(s) ds - \frac{\alpha}{(1-\alpha\eta) + \beta(1-\alpha)} \int_0^\eta (\eta-s)y(s) ds. \quad (2.6)$$

Then Lemma 2.1 is proved.

LEMMA 2.2. Suppose  $(1 - \alpha\eta) + \beta(1 - \alpha) \neq 0$ , then the Green's function for the BVP

$$-u'' = 0, \quad 0 \leq t \leq 1, \quad (2.7)$$

$$u(0) - \beta u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad (2.8)$$

is given by

$$G(t, s) = \begin{cases} \frac{(s + \beta)[(1 - t) - \alpha(\eta - t)]}{(1 - \alpha\eta) + \beta(1 - \alpha)}, & s \leq t, \quad s \leq \eta; \\ \frac{(s + \beta)(1 - t) + \alpha(t - s)(\eta + \beta)}{(1 - \alpha\eta) + \beta(1 - \alpha)}, & \eta < s \leq t; \\ \frac{(t + \beta)[(1 - s) - \alpha(\eta - s)]}{(1 - \alpha\eta) + \beta(1 - \alpha)}, & t \leq s \leq \eta; \\ \frac{(t + \beta)(1 - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)}, & s \geq t, \quad s \geq \eta. \end{cases} \quad (2.9)$$

PROOF. If  $t \leq \eta$ , the unique solution (2.3) can be stated from Lemma 2.1

$$\begin{aligned} u(t) &= - \int_0^t (t - s)y(s) ds + \frac{t + \beta}{(1 - \alpha\eta) + \beta(1 - \alpha)} \int_0^1 (1 - s)y(s) ds \\ &\quad - \frac{\alpha(t + \beta)}{(1 - \alpha\eta) + \beta(1 - \alpha)} \int_0^\eta (\eta - s)y(s) ds \\ &= \int_0^t -(t - s)y(s) ds + \int_0^t \frac{(t + \beta)(1 - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds \\ &\quad + \int_t^\eta \frac{(t + \beta)(1 - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds + \int_\eta^1 \frac{(t + \beta)(1 - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds \\ &\quad + \int_0^t \frac{-\alpha(t + \beta)(\eta - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds + \int_t^\eta \frac{-\alpha(t + \beta)(\eta - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds \\ &= \int_0^t \frac{(s + \beta)[(1 - t) - \alpha(\eta - t)]}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds + \int_t^\eta \frac{(t + \beta)[(1 - s) - \alpha(\eta - s)]}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds \\ &\quad + \int_\eta^1 \frac{(t + \beta)(1 - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds. \end{aligned}$$

Similarly, if  $t \geq \eta$ , the unique solution (3.3) can be stated

$$\begin{aligned} u(t) &= \int_0^\eta \frac{(s + \beta)[(1 - t) - \alpha(\eta - t)]}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds + \int_\eta^t \frac{(s + \beta)(1 - t) + \alpha(t - s)(\eta + \beta)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds \\ &\quad + \int_t^1 \frac{(t + \beta)(1 - s)}{(1 - \alpha\eta) + \beta(1 - \alpha)} y(s) ds. \end{aligned}$$

Therefore, the unique solution of (2.1), (2.2) is  $u(t) = \int_0^1 G(t, s)y(s) ds$ . Lemma 2.2 is proved.

Let  $w(t) = \int_0^1 G(t, s)h(s) ds$ . Obviously,  $w(t)$  is the unique solution to BVP (2.1), (2.2) for  $y(t) = h(t)$ .

LEMMA 2.3. Let  $X = C[0, 1]$ ,  $K = \{u \in X : u(t) \geq 0\}$ . Suppose  $T : X \rightarrow X$  is completely continuous. Define  $\theta : TX \rightarrow K$  by

$$(\theta y)(t) = \max\{y(t), w(t)\}, \quad \text{for } y \in TX,$$

where  $w \in C^1[0, 1]$ ,  $w(t) \geq 0$  is a given function. Then

$$\theta \circ T : X \rightarrow K$$

is also a completely continuous operator.

PROOF. The complete continuity of  $T$  implies that  $T$  is continuous and maps each bounded subset in  $X$  to a relatively compact set. Denote  $\theta y$  by  $\bar{y}$ .

Given a function  $h \in C[0, 1]$ , for each  $\varepsilon > 0$  there is  $\delta > 0$  such that

$$\|Th - Tg\| < \varepsilon, \quad \text{for } g \in X, \quad \|g - h\| < \delta.$$

Since

$$\begin{aligned} |(\theta Th)(t) - (\theta Tg)(t)| &= |\max\{(Th)(t), w(t)\} - \max\{(Tg)(t), w(t)\}| \\ &\leq |(Th)(t) - (Tg)(t)| < \varepsilon, \end{aligned}$$

we have

$$\|(\theta T)h - (\theta T)g\| < \varepsilon, \quad \text{for } g \in X, \quad \|g - h\| < \delta,$$

and then  $\theta T$  is continuous.

For an arbitrarily given bounded set  $D \subset X$  and  $\forall \varepsilon > 0$ , there are  $y_i, i = 1, \dots, m$  such that

$$TD \subset \bigcup_{i=1}^m B(y_i, \varepsilon),$$

where  $B(y_i, \varepsilon) = \{u \in X : \|u - y_i\| < \varepsilon\}$ . Then,  $\forall \bar{y}(t) \in (\theta \circ T)D$ , there is  $y \in TD$  such that  $\bar{y}(t) = \max\{y(t), w(t)\}$ . We choose one  $y_i \in \{y_1, \dots, y_m\}$  such that  $\|y - y_i\| < \varepsilon$ . The fact

$$\max_{0 \leq t \leq 1} |\bar{y}(t) - \bar{y}_i(t)| \leq \max_{0 \leq t \leq 1} |y(t) - y_i(t)|$$

implies  $\bar{y} \in B(\bar{y}_i, \varepsilon)$ . Then  $(\theta \circ T)D$  has a finite  $\varepsilon$ -net, and therefore,  $(\theta \circ T)(D)$  is relatively compact.

### 3. EXISTENCE OF SOLUTION

Let  $X = C[0, 1]$  and  $K = \{u \in X : u(t) \geq 0\}$ . Denote by  $\|\cdot\|$  the supremum norm on  $X$ .

Suppose the following conditions are satisfied:

(H1)  $\alpha, \beta > 0, 0 < \eta < 1, \alpha < 1$ ;

(H2)  $f : [0, 1] \times [0, \infty) \rightarrow R$  is continuous;

(H3)  $h(t)$  is a nonnegative measurable function on  $[0, 1], 0 < \int_0^1 h(t) dt < \infty$ .

If (H1) holds, then  $(1 - \alpha\eta) + \beta(1 - \alpha) > 0$ . We claim  $G(t, s) \geq 0$ , here  $G(t, s)$  given in (2.9) is the Green function for (2.7), (2.8). In fact, the conclusion is trivial in the case that  $\eta < s \leq t$  or  $s \geq t, s \geq \eta$ . If  $s \leq t, s \leq \eta$ , we get that

$$G(t, s) = \frac{(s + \beta)[(1 - t) - \alpha(\eta - t)]}{(1 - \alpha\eta) + \beta(1 - \alpha)} \geq 0$$

by reason of  $1 - t \geq \alpha(\eta - t)$ . Similarly, in view to  $1 - s \geq \alpha(\eta - s)$ , we have  $G(t, s) \geq 0$  for  $t \leq s \leq \eta$ . So  $G(t, s) \geq 0$  in all cases. Moreover,  $u(t) = \int_0^1 G(t, s)y(s) ds$ , here  $u(t)$  is the unique solution of BVP (2.1), (2.2). Then for  $y(t) \equiv 1$ , we have

$$\begin{aligned} \int_0^1 G(t, s) ds &= - \int_0^t (t - s) ds + \frac{t + \beta}{(1 - \alpha\eta) + \beta(1 - \alpha)} \int_0^1 (1 - s) ds \\ &\quad - \frac{\alpha(t + \beta)}{(1 - \alpha\eta) + \beta(1 - \alpha)} \int_0^\eta (\eta - s) ds \\ &= -\frac{1}{2}t^2 + \frac{(t + \beta)(1 - \alpha\eta^2)}{2[(1 - \alpha\eta) + \beta(1 - \alpha)]} < \infty, \quad t \in [0, 1]. \end{aligned}$$

By the Hölder inequality, we have  $\int_0^1 |G(t, s)h(s)| ds \leq (\int_0^1 |G(t, s)|^2)^{1/2} (\int_0^1 |h(s)|^2)^{1/2} < \infty$ ,  $t \in [0, 1]$ . Let  $A = \max_{0 \leq t \leq 1} \int_0^1 G(t, s)h(s) ds$ . We suppose (H1)–(H3) hold throughout the paper.

THEOREM 3.1. Suppose there are  $r > M > 0$  such that

$$0 < \frac{M}{\min_{0 \leq t \leq 1} f(t, Mw(t))} = a \leq b = \frac{r}{A \max_{\substack{0 \leq t \leq 1 \\ Mw(t) \leq u \leq r}} f(t, u)}. \quad (3.1)$$

Then BVP (1.5), (1.6) has at least one positive solution  $u_1(t)$  satisfying

$$0 < Mw(t) \leq u_1(t), \quad 0 < t < 1 \quad \text{and} \quad \|u_1\| \leq r,$$

if  $\lambda \in [a, b]$ .

PROOF. Let

$$f^*(t, u) = \begin{cases} f(t, u), & u \geq Mw(t), \\ f(t, Mw(t)), & u \leq Mw(t), \end{cases} \quad (3.2)$$

and define  $T : K \rightarrow X$  by

$$(Tu)(t) = \lambda \int_0^1 G(t, s) h(s) f^*(s, u(s)) ds, \quad 0 \leq t \leq 1. \quad (3.3)$$

Then  $T$  is on  $K$  a completely continuous operator. For the operator  $\theta : X \rightarrow K$  defined by

$$(\theta u)(t) = \max\{u(t), 0\}, \quad (3.4)$$

Lemma 2.2 implies that  $\theta \circ T : K \rightarrow K$  is also completely continuous.

Take  $\Omega = \{u \in K : \|u\| < r\}$ . Given  $u \in \partial\Omega$ , set  $I = \{t \in [0, 1] : f^*(t, u(t)) \geq 0\}$ . Then

$$\begin{aligned} (\theta \circ T)u(t) &= \max \left\{ \lambda \int_0^1 G(t, s) h(s) f^*(s, u(s)) ds, 0 \right\} \\ &\leq \lambda \int_I G(t, s) h(s) f^*(s, u(s)) ds \\ &\leq b \max_{\substack{0 \leq t \leq 1 \\ 0 \leq u \leq r}} f^*(t, u) \int_I G(t, s) h(s) ds \\ &\leq Ab \max_{\substack{0 \leq t \leq 1 \\ Mw(t) \leq u \leq r}} f(t, u) \\ &\leq r. \end{aligned}$$

If there is a  $u \in \partial\Omega$  such that  $(\theta \circ T)u = u$ , then  $\theta \circ T$  has a fixed point in  $\bar{\Omega}$ . Suppose for  $\forall u \in \partial\Omega$ ,  $(\theta \circ T)u \neq u$ , it follows that

$$\deg_K \{I - \theta \circ T, \Omega, 0\} = 1,$$

where  $\deg_K$  stands for the degree on cone  $K$ . Then  $\theta \circ T$  has a fixed point in  $\Omega$ . So in both the cases  $\theta \circ T$  has a fixed point  $u_1$  in  $\bar{\Omega}$ .

We claim that

$$(Tu_1)(t) \geq Mw(t), \quad t \in [0, 1]. \quad (3.5)$$

Otherwise, there is  $t_0 \in [0, 1]$  such that

$$Mw(t_0) - (Tu_1)(t_0) = \max_{0 \leq t \leq 1} \{Mw(t) - (Tu_1)(t)\} = L > 0. \quad (3.6)$$

Now we prove  $t_0 \in (0, 1)$ . Suppose the contrary, if  $t_0 = 0$ , then  $Mw'(0) - (Tu_1)'(0) \leq 0$ . Since both  $Mw(t)$  and  $(Tu_1)(t)$  satisfy the boundary condition (1.6), we have

$$[Mw(0) - (Tu_1)(0)] - \beta [Mw'(0) - (Tu_1)'(0)] = 0,$$

which contradicts (H1). If  $t_0 = 1$ , similarly it contradicts (1.6),

$$Mw(1) - Tu_1(1) = \alpha[Mw(\eta) - Tu_1(\eta)].$$

So  $t_0 \in (0, 1)$ , and

$$Mw'(t_0) - (Tu_1)'(t_0) = 0.$$

We prove now

$$Mw(t) > Tu_1(t), \quad t \in [0, 1]. \quad (3.7)$$

Otherwise, there exist  $t_1 \in [0, t_0] \cup (t_0, 1]$  such that

$$Mw(t_1) - Tu_1(t_1) = 0 \quad \text{and} \quad Mw(t) - Tu_1(t) > 0, \quad t \in (t_1, t_0] \quad \text{or} \quad t \in [t_0, t_1].$$

Without loss of generality, we suppose  $t_1 \in [0, t_0]$ . Then for  $t \in (t_1, t_0]$ ,

$$\begin{aligned} Mw'(t) - (Tu_1)'(t) &= Mw'(t_0) - (Tu_1)'(t_0) - \int_t^{t_0} [Mw'(s) - (Tu_1)'(s)]' ds \\ &= \int_t^{t_0} [M - \lambda f^*(s, u_1(s))] ds \\ &\leq 0, \end{aligned}$$

i.e.,  $Mw'(t) - (Tu_1)'(t) \leq 0$ , and then

$$Mw(t_0) - Tu_1(t_0) \leq Mw(t_1) - Tu_1(t_1) = 0,$$

a contradiction to (3.6). So (3.7) holds.

However,

$$\begin{aligned} Mw(t_0) - (Tu_1)(t_0) &= \int_0^1 G(t_0, s)h(s)Mds - \lambda \int_0^1 G(t_0, s)h(s)f^*(s, u_1(s)) ds \\ &= \int_0^1 G(t_0, s)h(s) [M - \lambda f^*(s, u_1(s))] ds \\ &\leq \left[ M - a \min_{0 \leq t \leq 1} f(t, Mw(t)) \right] \int_0^1 G(t_0, s)h(s) ds \\ &= 0, \end{aligned}$$

a contradiction to (3.6). So (3.5) holds. Then  $(\theta \circ T)u_1 = Tu_1 = u_1$  and  $u_1(t)$  is a solution of BVP (1.5), (1.6).

**THEOREM 3.2.** Suppose  $f(t, 0) \geq 0$ ,  $h(t)f(t, 0) \not\equiv 0$ , and there is  $r > 0$  such that

$$b = \frac{r}{A \max_{\substack{0 \leq t \leq 1 \\ 0 \leq u \leq r}} f(t, u)} > 0. \quad (3.8)$$

Then when  $\lambda \leq b$ , BVP (1.5), (1.6) has at least one positive solution  $u_1(t)$  satisfying

$$0 < \|u_1\| \leq r.$$

**PROOF.** Let

$$f^*(t, u) = \begin{cases} f(t, u), & u \geq 0, \\ f(t, 0) - u, & u < 0. \end{cases} \quad (3.9)$$

In a similar way, we can get the conclusion.

COROLLARY 3.1. Suppose there is  $M > 0$  such that

$$a = \frac{M}{\min_{0 \leq t \leq 1} f(t, Mw(t))} > 0 \quad (3.10)$$

and

$$\lim_{u \rightarrow \infty} \frac{\max_{0 \leq t \leq 1} f(t, u)}{u} \leq 0, \quad (3.11)$$

then for  $\lambda \geq a$ , BVP (1.5),(1.6) has at least a positive solution  $u_1$  with

$$0 < Mw(t) \leq u_1(t), \quad \|u_1\| < \infty.$$

PROOF. It suffices to show that for  $\forall b > a$ , there is  $r > 0$  such that

$$b \leq \frac{r}{A \max_{\substack{0 \leq t \leq 1 \\ Mw(t) \leq u \leq r}} f(t, u)}. \quad (3.12)$$

Fix  $b > a > 0$ . Condition (3.11) implies that there is  $L > 0$  such that

$$\frac{\max_{0 \leq t \leq 1} f(t, u)}{u} < \frac{1}{bA}, \quad \text{for } u \geq L,$$

and there exists  $r > L$  such that

$$\frac{\max_{\substack{0 \leq t \leq 1 \\ Mw(t) \leq u \leq L}} f(t, u)}{r} < \frac{1}{bA}.$$

Hence,

$$\begin{aligned} \frac{\max_{\substack{0 \leq t \leq 1 \\ Mw(t) \leq u \leq r}} f(t, u)}{r} &\leq \max \left\{ \frac{\max_{\substack{0 \leq t \leq 1 \\ Mw(t) \leq u \leq L}} f(t, u)}{r}, \frac{\max_{\substack{0 \leq t \leq 1 \\ L \leq u \leq r}} f(t, u)}{r} \right\} \\ &< \max \left\{ \frac{1}{bA}, \max_{L \leq u \leq r} \left[ \frac{\max_{0 \leq t \leq 1} f(t, u)}{u} \right] \right\} \\ &< \frac{1}{bA}, \end{aligned}$$

and in turn (3.12) holds. Applying Theorem 3.1, we proved this corollary since  $b > a$  is arbitrary.

COROLLARY 3.2. Suppose condition (3.11) holds and

$$f(t, 0) \geq 0, \quad h(t)f(t, 0) \neq 0, \quad t \in (0, 1),$$

then for  $\forall \lambda \in R$ , BVP (1.5),(1.6) has at least a positive solution  $u_1$  with  $0 < \|u_1\| < \infty$ .

PROOF. Condition (3.8) can be deduced from (3.11) for  $\forall b > 0$ ,  $a$  in Corollary 3.1. Then Theorem 3.2 implies this corollary.

REMARK 3.1. Theorem 3.1 and Corollary 3.1 can be applied to the case  $f \in C([0, 1] \times (0, \infty), R)$ , i.e.,  $f$  is singular at  $u = 0$ .

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